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SOUND PROPAGATION IN THE SEA--RAY TRACING, CURVATURE CORRECTION--ETC(U)

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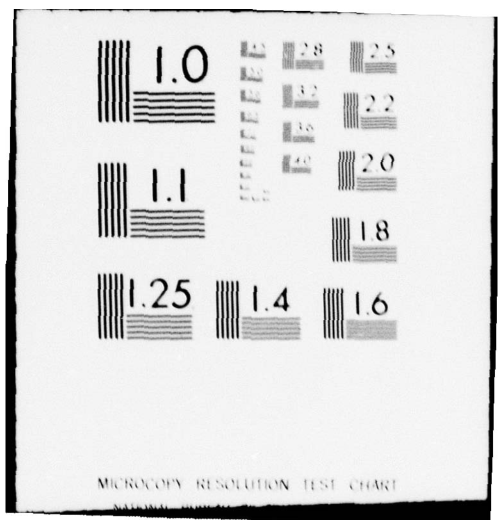
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LEVEL III

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Sound Propagation in the Sea -- Ray Tracing.
CURVATURE CORRECTIONS FOR THE EARTH (SEA LEVEL) SHAPED AS AN ELLIPSOID OF REVOLUTION.

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For the Officer-in-Charge
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Sound Propagation in the Sea -- Ray Tracing

CURVATURE CORRECTIONS FOR THE EARTH (SEA LEVEL) SHAPED AS AN ELLIPSOID OF REVOLUTION

by: M. M. Holl

ABSTRACT

1. The Problem

The three-dimensional tracing of a sound ray in the sea may be referred to the surface coordinate system of *latitude and longitude* and to the depth below the sea surface. This coordinate system involves curvature terms.

phi

~~For~~ the earth shaped as an ellipsoid of revolution ~~we~~ may be referred to two definitions of latitude: the geocentric latitude ϕ and the astronomical latitude Φ. These are illustrated in Fig. 2. There is a one-to-one relationship between these definitions and the transformation is simple as will be shown. ~~We choose to use~~ the geocentric latitude ϕ WAS CHOSEN as principal latitude coordinate.

PHI

THETA

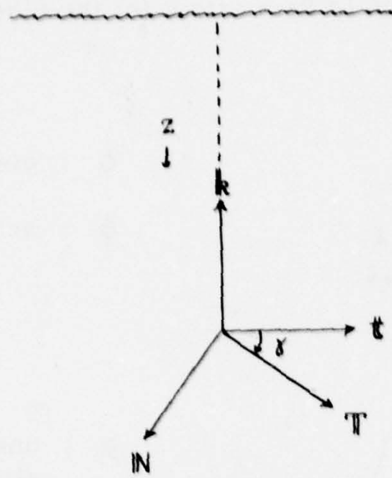
phi

Let the direction of propagation of a sound ray at an arbitrary point in its progress -- latitude ϕ, longitude θ and depth z -- be defined by the unit vector $\mathbf{T}$. It makes the angle γ with the horizontal as shown in Fig. 1. The horizontal component of the unit vector $\mathbf{T}$ defines the direction of the unit vector $\mathbf{t}$.

gamma

T

ABSTRACT



$$\mathbb{T} = \mathbb{t}\cos\gamma - \mathbb{k}\sin\gamma$$

$$\mathbb{N} = -\mathbb{t}\sin\gamma - \mathbb{k}\cos\gamma$$

Fig. 1 Definition of Parameters in the Vertical Plane of the Ray

In Fig. 2 we define and illustrate geometric parameters which are relevant to the problem. These include the angle β which $\mathbb{t}$ makes with the unit vector $\mathbb{i}$: $\mathbb{t}$ is directed β radians north of due east.

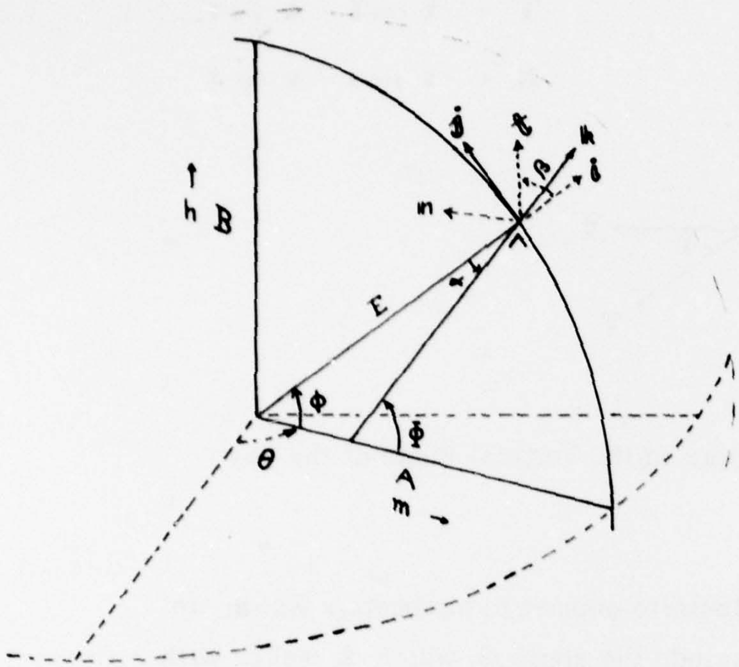
The coordinate-curvature terms arise in prescribing the ray direction $\mathbb{T}$ by the angles γ and β. The angle γ is referred to the horizontal which itself is turning in space from location to location. The angle β is referred to the east which itself is turning from location to location.

We may formulate the curvature terms directly. The reference directions turn only by horizontal progress and the angles are defined in specific planes. The terms are

$$(\partial\gamma/\partial s)_c = + \mathbb{t}\cdot\nabla\mathbb{t}\cdot\mathbb{k} \tag{1}$$

$$(\partial\beta/\partial s)_c = - \mathbb{t}\cdot\nabla\mathbb{i}\cdot\mathbb{j} \tag{2}$$

Fig. 2 Definition of Parameters for the Surface



ϕ : geocentric latitude

Φ : astronomical latitude

$$\alpha = \Phi - \phi$$

$\mathbb{k}$: unit vector, surface normal, alligned with local gravity, subtends angle with equatorial plane as shown.

$\mathbb{i}$: unit vector, surface tangent, directed eastward.

$\mathbb{j}$: unit vector, surface tangent, directed northward.

$$\mathbb{i} \times \mathbb{j} = \mathbb{k}$$

A : equatorial radius of earth $\quad A > B$
B : polar radius of earth

E : radius at geocentric latitude ϕ (astronomical latitude Φ)

m,h : Cartesian coordinates in arbitrary meridional plane of longitude θ.

Solid lines lie in the meridional plane; dashed lines do not.

$\mathbb{t}$: unit vector, tangent to surface trace.
$\mathbb{n}$: unit vector, surface tangent, normal to $\mathbb{t}$.

$$\mathbb{k} \times \mathbb{t} = \mathbb{n}$$

$$\mathbb{t} = \cos\beta \; \mathbb{i} + \sin\beta \; \mathbb{j}$$

$$\mathbb{n} = -\sin\beta \; \mathbb{i} + \cos\beta \; \mathbb{j}$$

2. <u>Geometry of the Surface</u>

The intersect of the arbitrary meridional plane and the surface is an ellipse:

$$\frac{h^2}{B^2} + \frac{m^2}{A^2} = 1 \tag{3}$$

We substitute

$$\begin{aligned} h &= E \sin \phi \\ m &= E \cos \phi \end{aligned} \tag{4}$$

and obtain

$$E^2 = \frac{A^2 \quad B^2}{A^2 \sin^2\phi + B^2 \cos^2\phi} \tag{5}$$

We also note that

$$\tan \phi = \frac{h}{m} \tag{6}$$

An incremental northward displacement on the ellipse may be expressed by

$$h \frac{\delta h}{B^2} + m \frac{\delta m}{A^2} = 0 \tag{7}$$

where δh is positive and δm is negative. This displacement is directed along $\mathbf{j}$. Hence

$$\tan \Phi = - \frac{\delta m}{\delta h} = \frac{A^2}{B^2} \frac{h}{m}$$

$$= \frac{A^2}{B^2} \tan \phi \qquad (8)$$

This is the relationship between the astronomical latitude Φ and the geocentric latitude ϕ. Their difference angle

$$\alpha = \Phi - \phi \qquad (9)$$

is given by

$$\tan \alpha = \tan (\Phi - \phi) = \frac{\tan \Phi - \tan \phi}{1 + \tan \Phi \tan \phi}$$

$$= \frac{(A^2 - B^2) \tan \phi}{B^2 + A^2 \tan^2 \phi} \qquad (10)$$

We also require the ratio of latitude increments: Differentiation of Eq. (8) yields

$$\sec^2 \Phi \quad \delta \Phi = \frac{A^2}{B^2} \sec^2 \phi \quad \delta \phi$$

$$\frac{\delta \Phi}{\delta \phi} = \frac{A^2}{B^2} \frac{\cos^2 \Phi}{\cos^2 \phi}$$

$$= \frac{A^2 B^2}{\cos^2 \phi \, (B^4 + A^4 \tan^2 \phi)} \tag{11}$$

3. Curvatures of the Coordinate Unit Vectors

We are now in position to express and develop the curvatures of the coordinate unit vectors. These are expressed by an incremental directed turning of the unit vector divided by the space increment, in the pertinent direction, over which this turning takes place:

$$\mathbf{i} \cdot \nabla \mathbf{i} = \frac{\cancel{\delta \theta}}{E \cos \phi \, \cancel{\delta \theta}} \left(\mathbf{j} \sin \Phi - \mathbf{k} \cos \Phi \right) \tag{12}$$

$$\mathbf{i} \cdot \nabla \mathbf{j} = - \frac{\mathbf{i} \sin \Phi \, \cancel{\delta \theta}}{E \cos \phi \, \cancel{\delta \theta}} \tag{13}$$

$$\mathbf{i} \cdot \nabla \mathbf{k} = \frac{\mathbf{i} \cos \Phi \, \cancel{\delta \theta}}{E \cos \phi \, \cancel{\delta \theta}} \tag{14}$$

$$\mathbf{j} \cdot \nabla \mathbf{i} = 0 \tag{15}$$

$$\mathbf{j} \cdot \nabla \mathbf{j} = - \frac{\mathbf{k} \, \delta \Phi}{E \sec \alpha \, \delta \phi} \tag{16}$$

$$\mathbf{j} \cdot \nabla \mathbf{k} = \frac{\mathbf{j} \quad \delta \Phi}{E \sec \alpha \quad \delta \phi} \tag{17}$$

$$\mathbf{k} \cdot \nabla \mathbf{i} = \mathbf{k} \cdot \nabla \mathbf{j} = \mathbf{k} \cdot \nabla \mathbf{k} = 0 \tag{18}$$

In Eqs. (16) and (17) we substitute from Eq. (11):

$$\mathbf{j} \cdot \nabla \mathbf{j} = - \frac{\mathbf{k} \quad \cos \alpha \quad A^2 B^2}{E \cos^2 \phi \ (B^4 + A^4 \tan^2 \phi)} \tag{19}$$

$$\mathbf{j} \cdot \nabla \mathbf{k} = \frac{\mathbf{j} \quad \cos \alpha \quad A^2 B^2}{E \cos^2 \phi \ (B^4 + A^4 \tan^2 \phi)} \tag{20}$$

4. Development of the Curvature Terms

Equation (1) is developed as follows

$$\begin{aligned}
(\partial \gamma / \partial s)_c &= \mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{k} \\
&= (\cos\beta \ \mathbf{i} + \sin\beta \ \mathbf{j}) \cdot \nabla (\cos\beta \ \mathbf{i} + \sin\beta \ \mathbf{j}) \cdot \mathbf{k} \\
&= \cos^2\beta \ \mathbf{i} \cdot \nabla \mathbf{i} \cdot \mathbf{k} + \cos\beta \ \sin\beta \ \mathbf{i} \cdot \nabla \mathbf{j} \cdot \mathbf{k} \\
&\quad + \sin\beta \ \cos\beta \ \mathbf{j} \cdot \nabla \mathbf{i} \cdot \mathbf{k} + \sin^2\beta \ \mathbf{j} \cdot \nabla \mathbf{j} \cdot \mathbf{k} \\
&= - \frac{\cos^2\beta \ \cos \Phi}{E \cos \phi} - \frac{\sin^2\beta \ \cos\alpha \ A^2 B^2}{E \cos^2\phi \ (B^4 + A^4 \tan^2 \phi)}
\end{aligned} \tag{21}$$

Equation (2) is developed as follows

$$(\partial \beta / \partial s)_c = - \mathbf{t} \cdot \nabla \mathbf{i} \cdot \mathbf{j}$$

$$= - (\cos \beta\, \mathbf{i} + \sin \beta\, \mathbf{j}) \cdot \nabla \mathbf{i} \cdot \mathbf{j}$$

$$= - \cos \beta\, \mathbf{i} \cdot \nabla \mathbf{i} \cdot \mathbf{j} - \sin \beta\, \mathbf{j} \cdot \nabla \mathbf{i} \cdot \mathbf{j}$$

$$= - \frac{\cos \beta \sin \Phi}{E \cos \phi} \qquad (22)$$

We now wish to eliminate E, Φ and α by substitution in terms of ϕ according to Eqs. (5), (8) and (10). For abbreviation, where convenient, we adapt

$$K \equiv \tan \phi \quad ; \qquad F \equiv \frac{A^2}{B^2} \qquad (23)$$

We may rewrite Eqs. (5), (8) and (10) as follows:

$$E^2 = \frac{A^2 \; (1 + K^2)}{(1 + FK^2)} \qquad (24)$$

$$\tan \Phi = FK \qquad (25)$$

$$\tan \alpha = \frac{(F - 1)\, K}{(1 + FK^2)} \qquad (26)$$

It follows that

$$\cos \Phi = \frac{1}{(1 + F^2 K^2)^{1/2}} \tag{27}$$

$$\sin \Phi = \frac{FK}{(1 + F^2 K^2)^{1/2}} \tag{28}$$

$$\cos \alpha = \frac{(1 + FK^2) \cos \phi}{(1 + F^2 K^2)^{1/2}} \tag{29}$$

With the appropriate substitutions, Eqs. (21) and (22) become

$$(\partial \gamma / \partial s)_c = - \frac{1}{A} \left\{ \frac{1 + FK^2}{1 + F^2 K^2} \right\}^{1/2} \frac{\cos^2 \beta + F \sin^2 \beta + F^2 K^2}{1 + F^2 K^2} \tag{30}$$

$$(\partial \beta / \partial s)_c = - \frac{\cos \beta}{A} \; FK \left\{ \frac{1 + FK^2}{1 + F^2 K^2} \right\}^{1/2} \tag{31}$$

These are the coordinate-curvature terms, on γ and β, as functions of the geocentric latitude ϕ and the ray-direction angle β.

In the special limit of a sphere:

$$A = B \equiv R; \qquad \phi = \Phi; \qquad F = 1$$

the curvature terms simplify to

$$(\partial \gamma / \partial s)_{C,R} = -\frac{1}{R} \tag{32}$$

$$(\partial \beta / \partial s)_{C,R} = -\frac{\cos\beta \tan\Phi}{R} \tag{33}$$

5. The Ray Tracing Equations

The ray tracing equation, in vector form, may be expressed by

$$\mathbb{T} \cdot \nabla \mathbb{T} = \mathbb{T} \times (\mathbb{T} \times \nabla \ln C) \tag{34}$$

where $\mathbb{T}$ is the unit-vector ray direction and C is the sound speed. We take the components along $\mathbb{N}$ and $\mathbb{n}$ -- unit vectors defined in Figs. (1) and (2) respectively -- and obtain

$$\mathbb{T} \cdot \nabla \mathbb{T} \cdot \mathbb{N} = -\mathbb{N} \cdot \nabla \ln C \tag{35}$$

$$\mathbb{T} \cdot \nabla \mathbb{T} \cdot \mathbb{n} = -\mathbb{n} \cdot \nabla \ln C \tag{36}$$

The left-hand sides may be transformed by geometrical consideration:

$$\mathbb{T} \cdot \nabla \mathbb{T} \cdot \mathbb{N} = \partial \gamma / \partial S - \cos\gamma \; \mathbb{t} \cdot \nabla \mathbb{t} \cdot \mathbb{k} \tag{37}$$

$$\mathbb{T} \cdot \nabla \mathbb{T} \cdot \mathbb{n} = \cos\gamma \left\{ \partial \beta / \partial S + \cos\gamma \; \mathbb{t} \cdot \nabla \mathbb{i} \cdot \mathbb{j} \right\} \tag{38}$$

where S is a linear measure along the ray trace:

$$\delta s = \delta S \cos\gamma \tag{39}$$

The coordinate-curvature terms are expressed by Eqs. (1) and (2) and the ray-tracing equations may be written

$$\frac{\partial \gamma}{\partial s} = \left(\frac{\partial \gamma}{\partial s}\right)_c - \sec \gamma \quad \mathbb{N} \cdot \nabla \ln C \tag{40}$$

$$\frac{\partial \beta}{\partial s} = \left(\frac{\partial \beta}{\partial s}\right)_c - \sec^2 \gamma \quad \mathbb{n} \cdot \nabla \ln C \tag{41}$$

The curvature terms are given by Eqs. (30) and (31).

6. Linearization as to Oblateness of Earth

If we define ϵ by

$$A \equiv (1 + \epsilon) B \tag{42}$$

where A is the equatorial radius and B is the polar radius of the earth, then

$$0 < \epsilon << 1 , \tag{43}$$

and we may linearize Eqs. (30) and (31) by dismissing squares and higher powers of ϵ as negligible. In this linearization

$$A^2 = (1 + 2\epsilon) B^2 \tag{44}$$

$$F = (1 + 2\epsilon) \tag{45}$$

$$F^2 = (1 + 4\epsilon) \tag{46}$$

We also define as the earth's mean radius

$$R \equiv \frac{1}{2}(A + B) \tag{47}$$

It follows that

$$A = (1 + \frac{1}{2}\epsilon)\ R \tag{48}$$

With these substitutions we linearize Eqs. (30) and (31) and obtain

$$\left(\frac{\partial \gamma}{\partial s}\right)_c = -\frac{1}{R}\left\{1 + \epsilon\ (2 \sin^2\beta\ \cos^2\phi\ - \sin^2\phi\ - \frac{1}{2})\right\} \tag{49}$$

$$\left(\frac{\partial \beta}{\partial s}\right)_c = -\frac{\cos\beta\ \tan\phi}{R}\left\{1 + \epsilon\ (\frac{1}{2} + \cos^2\phi)\right\} \tag{50}$$

We note that as $\epsilon \rightarrow 0$ we arrive at Eqs. (32) and (33) respectively, for the sphere; this, however, is not a check on the linear terms.

It is convenient, at this point, to note that the effect of the depth of the ray on the curvature terms can be taken into account by replacing R by R - z. This is equivalent to replacing A and B by A - z and B - z in all earlier expressions of the curvature terms; the effect on ϵ is negligible. In general the depth correction is smaller than the oblateness correction.

For convenience we also exhibit Eqs. (49) and (50) in terms of the astronomical latitude, Φ :

$$\left(\frac{\partial \gamma}{\partial s}\right)_c = -\frac{1}{R}\left\{1 + \epsilon\left(2 \sin^2\beta \cos^2\Phi - \sin^2\Phi - \frac{1}{2}\right)\right\} \qquad (51)$$

$$\left(\frac{\partial \beta}{\partial s}\right)_c = -\frac{\cos\beta \tan\Phi}{R}\left\{1 - \epsilon\left(\frac{1}{2} + \sin^2\Phi\right)\right\} \qquad (52)$$

By linearization of Eq. (26) we find that the difference between the astronomical latitude Φ and the geocentric latitude ϕ is

$$\alpha = \Phi - \phi = \epsilon \sin 2\phi$$

We note that

$$\alpha = \sin\alpha = \tan\alpha = \epsilon \sin 2\phi = \epsilon \sin 2\Phi \qquad (53)$$

$$\cos\alpha = \sec\alpha = 1 \qquad (54)$$

The angle α is a maximum at about $\phi = \pm 45^{\circ}$.

7. Magnitude of Corrections for Oblateness

The oblateness factor does not enter in sea-surface or sea-bottom reflections. These reflections are treated in the same way as in the spherical case; no oblateness correction enters; this is because the angle γ is defined relative to the local vertical (see Fig. 1).

The values

$$A = 6.378 \times 10^6 \text{ meters}$$

$$B = 6.357 \times 10^6 \text{ meters} \tag{55}$$

have been quoted; perhaps better values are now available. These values give

$$\epsilon = \frac{A}{B} - 1 \approx 0.0033 \approx \frac{1}{300} \tag{56}$$

The oblateness corrections have the following ranges in the curvature terms:

$$\left(\frac{\partial \gamma}{\partial s}\right)_c = -\frac{1}{R}\left\{1 + (-0.005 \text{ to } +0.005)\right\} \tag{57}$$

for both, Eqs. (49) and (51).

$$\left(\frac{\partial \beta}{\partial s}\right)_c = -\frac{\cos\beta \tan\Phi}{R}\left\{1 + (0.0017 \text{ to } 0.005)\right\} \tag{58}$$

$$\left(\frac{\partial \beta}{\partial s}\right)_c = -\frac{\cos\beta \tan\Phi}{R}\left\{1 + (-0.005 \text{ to } -0.0017)\right\} \tag{59}$$

for Eqs. (50) and (52), respectively.

In moving northward at the equator, we have

$$(\partial \gamma / \partial s)_c = -\frac{1}{R}(1 + 0.005) \tag{60}$$

In a northward displacement of

$$\delta s = 0.1 \, R \approx 637{,}000 \text{ meters} \tag{61}$$

the oblateness correction amounts to 0.0005 radians. Spreading this correction linearly over the displacement results in a depth correction of

$$637{,}000 \times 0.00025 \approx 160 \text{ meters} \tag{62}$$

This is an upper-limit approximation to the displacement errors made in neglecting the earth's oblateness.

We also note from Eq. (53) that maximum α (at $\phi = \pm 45^{\circ}$) is

$$0.0033 \text{ radians} \approx 11 \text{ minutes} \tag{63}$$

8. <u>The Geodesic: Surface Ray with no Refraction</u>

Consider a ray which is trapped in the sea surface by continuous glancing reflection, but undergoing no refraction. Its trace describes a geodesic:

$$\mathbf{t} \cdot \nabla \mathbf{t} = \mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{k}\,\mathbf{k} \tag{64}$$

Its geodesic curvature (i.e. curvature in the surface) is zero:

$$\mathbf{t} \cdot \nabla \mathbf{t} \cdot \mathbf{n} = 0 \tag{65}$$

We have seen that Eq. (65) transforms into

$$\partial \beta / \partial s = (\partial \beta / \partial s)_c \tag{66}$$

We choose Eq. (22) to write

$$\frac{\partial \beta}{\partial s} = - \frac{\cos \beta \sin \Phi}{E \cos \phi} \tag{67}$$

for the governing equation for a geodesic.

The governing equation can be integrated once and obtains that, for a geodesic, β is a function of latitude only. We achieve this by substituting

$$\sin \beta \;\; \delta s = E \sec \alpha \;\; \delta \phi \tag{68}$$

as seen from Fig. 2, into Eq. (67) to eliminate δs. We obtain

$$\tan \beta \;\; \delta \beta = - \frac{\sin \Phi}{\cos \phi \;\; \cos \alpha} \;\; \delta \phi \tag{69}$$

We transform Eq. (28) to obtain

$$\sin \Phi = \sin \phi \frac{F}{\left\{ 1 + (F^2 - 1) \sin^2 \phi \right\}^{1/2}} \tag{70}$$

We transform Eq. (29) to obtain

$$\cos \alpha = \frac{1 + (F - 1) \;\; \sin^2 \phi}{\left\{ 1 + (F^2 - 1) \sin^2 \phi \right\}^{1/2}} \tag{71}$$

Introduction of Eqs. (70) and (71) in (69) yields

$$\tan \beta \;\; \delta \beta = - \tan \phi \frac{F}{1 + (F - 1) \sin^2 \phi} \;\; \delta \phi \tag{72}$$

We integrate Eq. (72) from the equator, $\phi = 0$, where $\beta = \beta_o$, to an arbitrary latitude ϕ. The result is

$$\cos\beta = \cos\beta_o \,(1 + F\tan^2\phi)^{1/2} \tag{73}$$

The maximum latitude which this ray can attain is reached when $\beta = 0$:

$$\tan^2\phi_m = \frac{\tan^2\beta_o}{F} \tag{74}$$

The analytical equation for the geodesic requires one more integration. We first transform Eq. (73) to obtain

$$\tan\beta = \left\{ \frac{\sec^2\beta_o}{1 + F\tan^2\phi} - 1 \right\}^{1/2} \tag{75}$$

From the geometry of Fig. 2 we obtain

$$\tan\beta = \frac{\cancel{R}\sec\alpha \;\; \delta\phi}{\cancel{R}\cos\phi \;\; \delta\theta} \tag{76}$$

It follows that, for the geodesic,

$$\frac{d\phi}{d\theta} = \cos\phi\,\cos\alpha \left\{ \frac{\sec^2\beta_o}{1 + F\tan^2\phi} - 1 \right\}^{1/2} \tag{77}$$

Equation (77) may also be expressed by

$$\int_{\theta_o}^{\theta} d\theta = \int_{o}^{\phi} \frac{\sec\phi \ \sec\alpha \ (1 + F \tan^2\phi)^{1/2}}{(\tan^2 \beta_o - F \tan^2\phi)^{1/2}} \, d\phi \tag{78}$$

The geodesic trace is obtained by ϕ cycling back and forth between ϕ_m and $-\phi_m$, the range given by Eq. (74).

In the linearization of Eq. (78), as to oblateness, we must concern ourselves with the range of ϕ. We may immediately set $\sec\alpha$ equal to one; its departure is second order in ϵ. Again using $K \equiv \tan\phi$ we transform Eq. (78) into

$$\theta - \theta_o = \int_{o}^{K} \frac{(1 + F K^2)^{1/2} \quad dK}{(1 + K^2)^{1/2} \quad (\tan^2 \beta_o - FK^2)^{1/2}} \tag{79}$$

We change the variable of integration once more to

$$P \equiv F^{1/2} K \tag{80}$$

and obtain

$$\theta - \theta_o = \int_{o}^{P} \frac{F^{-1/2} \quad (1 + P^2)^{1/2} \quad dP}{(1 + F^{-1} P^2)^{1/2} \quad (\tan^2 \beta_o - P^2)^{1/2}} \tag{81}$$

Linearization, following $F = 1 + 2\epsilon$, yields

$$\theta - \theta_o = \int_o^P \frac{dP}{(\tan^2 \beta_o - P^2)^{1/2}} - \epsilon \int_o^P \frac{dP}{(1 + P^2)\,(\tan^2 \beta_o - P^2)^{1/2}} \tag{82}$$

where the range of P is $\pm \tan \beta_o$.

We introduce ω by

$$\sin \omega = P \cot \beta_o$$

$$\tan \omega = P \,(\tan^2 \beta_o - P^2)^{-1/2} \tag{83}$$

Equation (82) transforms into

$$\theta - \theta_o = \int_o^{\arcsin (P \cot \beta_o)} d\omega - \epsilon \int_o^{\arcsin (P \cot \beta_o)} \frac{d\omega}{1 + \tan^2 \beta_o \sin^2 \omega} \tag{84}$$

The first integral is solved; the second is transformed once more:

$$\theta - \theta_o = \arcsin (P \cot \beta_o) - \epsilon \int_o^{P \left\{ \tan^2 \beta_o - P^2 \right\}^{-1/2}} \frac{dy}{1 + \sec^2 \beta_o \, y^2} \tag{85}$$

where y replaces the dummy variable, $\tan \omega$.

The linearized solution for the geodisic is

$$\theta - \theta_o = \arcsin (P \cot \beta_o)$$

$$- \epsilon \cos \beta_o \arctan \left\{ \sec \beta_o P (\tan^2 \beta_o - P^2)^{-1/2} \right\} \quad (86)$$

where

$$P = (1 + \epsilon) \tan \phi \quad (87)$$

has the range $\pm \tan \beta_o$. The longitude at the maximum latitude is

$$\theta_m - \theta_o = \pi/2 \; (1 - \epsilon \cos \beta_o) \quad (88)$$

For $\epsilon = 0$ the surface becomes a sphere, and the geodesic is a great circle: Equation (86) reduces to

$$\sin (\theta - \theta_o) = \tan \phi \cot \beta_o \quad (89)$$

This result could have been obtained directly from the spherical trigonometry of a right spherical triangle.

In Fig. 3 we show the extent to which orbits of the geodisic precess. For

$$\epsilon = \frac{1}{300} \quad \text{and} \quad \beta_o = 45 \text{ degrees} \quad (90)$$

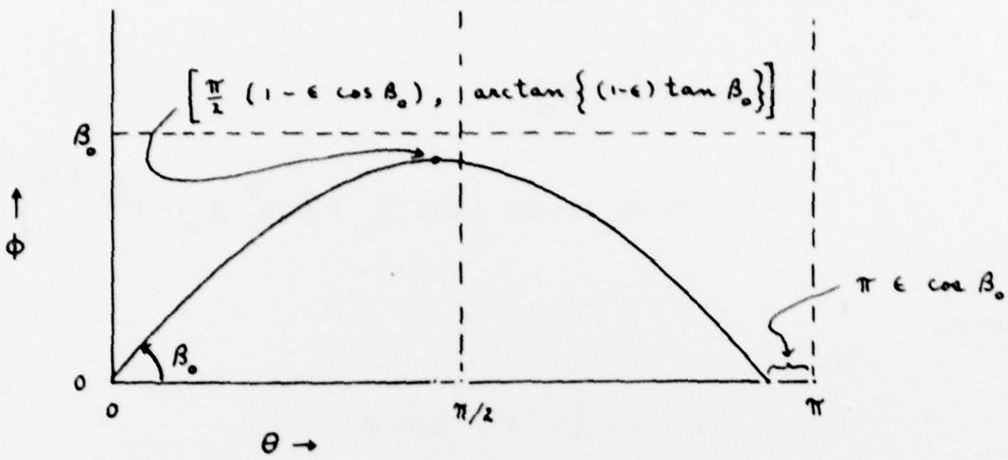


Fig. 3 Schematic of the Geodesic

the latitude maximum attained in Fig. 3 is located at

$$\theta = 89.7_9 \text{ degrees}$$

$$\phi = 44.9_0 \text{ degrees}$$

$$\Phi = 45.1_0 \text{ degrees} \tag{91}$$

The oblateness correction appears to be negligible within the practical limitations in sound-ray tracing in the sea.